PECULIAR LOCI OF AMPLE AND SPANNED LINE BUNDLES

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ABSTRACT. The *bad locus* and the *rude locus* of an ample and base point free linear system on a smooth complex projective variety are introduced and studied. Polarized surfaces of small degree, or whose degree is the square of a prime, with nonempty bad loci are completely classified. Several explicit examples are offered to describe the variety of behaviors of the two loci.

1. INTRODUCTION

Let X be a smooth complex projective variety of dimension $n \geq 2$ and let $L \in \operatorname{Pic}(X)$ be an ample line bundle spanned by a vector subspace $V \subseteq H^0(X, L)$.

This work is concerned with the study of two special loci associated with the pair (X, V) (see section 2 for notation).

Definition 1.1. The *bad locus* of (X, V) is the set

$$\mathcal{B}(X, V) = \{x \in X | \text{ every divisor in } |V - x| \text{ is reducible}\}.$$

A point $x \in \mathcal{B}(X, V)$ is called a *bad point* for (X, V).

Definition 1.2. The *rude locus* of (X, V) is the set

$$\mathcal{R}(X, V) = \{x \in X | |V - 2x| \neq \emptyset \text{ and every divisor in } |V - 2x| \text{ is reducible}\}.$$

A point $x \in \mathcal{R}(X, V)$ is called a *rude point* for (X, V).

The first aim of this work is to study $\mathcal{B}(X, V)$. This seems to be a very basic question, but to our knowledge it was not previously explicitly discussed in the literature. The phenomenon of bad points turns out, as a consequence of Bertini's second theorem, to be intrinsically two-dimensional, see Theorem 2.5, *i)*. Moreover the bad locus is a finite set, when nonempty, see Corollary 2.6. This last result follows from the relationship between the bad locus and the n-th jumping set of (X, V), as introduced in [12], defined as

$$\mathcal{J}_n(X, V) := \{x \in X \mid |V - x| = |V - 2x|\}.$$

Indeed it is $\mathcal{B}(X,V) \subseteq \mathcal{J}_n(X,L)$, see Theorem 2.5, *iv)*, and $\mathcal{J}_n(X,L)$ is known to be a finite set, [12], Theorem 1.2.

Although in most cases $\mathcal{B}(X,V)$ is actually empty, for example if $V = H^0(L)$ and L is very ample, see Corollary 2.6, or if (X,L) is a scroll, see Corollary 2.9, Section 3 contains concrete examples of surfaces for which $\mathcal{B}(X,V) = \mathcal{J}_2(X,V) \neq \emptyset$. These examples show that bad points indeed occur and that the result of Theorem 2.5, *iv)* is effective.

Most examples of nonempty bad loci that we present occur for surfaces of general type. Nonetheless a non empty $\mathcal{B}(X,V)$ is a very rare phenomenon. This suggests the problem of characterizing ample and spanned line bundles admitting bad points. An effective, complete classification of surfaces with non trivial bad locus and degree up to 11 or equal to p^2 where p is prime is presented in Section 4.

Our interest in *bad points* arose from the study of Seshadri constants $\epsilon(\mathcal{L}, x)$ of certain line bundles $\mathcal{L}$ on a surface S, which are slightly more general than ample and spanned ones. In particular we were interested in locating the points $x \in S$ where $\epsilon(\mathcal{L}, x)$ jumps. For an ample and spanned line bundle L on X one has $\epsilon(L, x) \geq 1$ for all $x \in X$, see for example [15] for $n = 2$. On the other hand one can easily see, Proposition 2.8, that $\epsilon(L, x) \geq 2$ if $x \in \mathcal{B}(X, L)$. This leads to the fact that if (X, L) is covered by lines then $\mathcal{B}(X, L) = \emptyset$, Corollary 2.9.

The second goal of this work is the study of the rude locus, as in Definition 1.2. The bad and the rude locus are intimately connected. It follows directly from the definitions that a bad point is also a rude point, but more precisely

$$(1) \qquad \mathcal{B}(X,V) = \mathcal{R}(X,V) \cap \mathcal{J}_n(X,V).$$

Notice that the above intersection is nonempty only if $n = 2$, as mentioned above, Theorem 2.5, *i)*. On the other hand there are many rude points that fail to be bad. Indeed, while $\mathcal{B}(X,V)$ is a finite set, the behavior of $\mathcal{R}(X,V)$ spans the whole gamut of possibilities. In section 5 examples are discussed in which $\mathcal{R}(X,L)$ is respectively empty (Example 5.4 for $n \geq 3$, Example 5.5 for $n \geq 4$), a finite set (Example 5.4 for $n = 2$), a divisor with a finite set removed (Example 5.12), a divisor (Example 5.2 for $s = \dim(|V|) = 2$, Example 5.11 for $n = 2$), a dense Zariski open subset (Example 5.5 for $n = 3$), the union of a dense Zariski open subset and a finite set (Example 5.5 for $n = 2$), the whole variety X. The remarkable phenomenon of having $\mathcal{R}(X,V) = X$ is addressed in Conjecture 6.1. Notice also that the relationship between $\mathcal{R}(X,V)$ and $\mathcal{B}(X,V)$ seen in (1) becomes extreme when (X,L) is a scroll, as it is $\mathcal{R}(X,L) = X$ while, as mentioned above, $\mathcal{B}(X,L) = \emptyset$.

2. Notation and Preliminary Results

Throughout this article X denotes a smooth, connected, projective variety of dimension n defined over the complex field $\mathbb{C}$. Sometimes we call such an X an n-fold. If $n = 2$ the variety is often denoted with S. Its structure sheaf is denoted by $\mathcal{O}_X$ and the canonical sheaf of holomorphic n-forms on X is denoted by K_X or simply K when the ambient variety is understood. Cartier divisors, their associated line bundles and the invertible sheaves of their holomorphic sections are used with no distinction. Mostly additive notation is used for their group. Given two divisors L and M we denote linear equivalence by $L \sim M$ and numerical equivalence by $L \equiv M$. For any coherent sheaf $\mathcal{F}$ on X, $h^i(X, \mathcal{F})$ is the complex dimension of $H^i(X, \mathcal{F})$. When the ambient variety is understood, we often write $H^i(\mathcal{F})$ and $h^i(\mathcal{F})$ respectively for $H^i(X, \mathcal{F})$ and $h^i(X, \mathcal{F})$. Let L be a line bundle on X. If L is ample, the pair (X, L) is called a *polarized variety*. For a subspace $V \subseteq H^0(X, L)$ the following notations are used:

$|V|$, the linear system associated with V;

$|V - rx|$, the linear system of divisors in $|V|$ passing trough a point $x \in X$ with multiplicity at least r;

$\mathrm{Bs}|V|$, the base locus of the linear system $|V|$;

φ_V, the rational map given by $|V|$;

If $V = H^0(L)$ we write L instead of V in all of the above. For a line bundle $L \not\simeq \mathcal{O}_X$ we say that $|L|$ is *free* if $\mathrm{Bs}|L| = \emptyset$ or equivalently if L is spanned, i.e. generated by its global sections;

The following notation and definitions are also used:

$g = g(X, L)$, the *sectional genus* of (X, L), defined by $2g - 2 = L^{n-1}(K_X + (n-1)L)$.

$\mathbb{F}_e$, the Segre-Hirzebruch surface of invariant e.

For the special polarized varieties arising in adjunction theory we adopt

the usual adjunction theoretic terminology, e.g. see [3]. In particular, in accordance with it, $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))$ is not a scroll.

When S is a $\mathbb{P}^1$-bundle over a curve with fundamental section C_0 and generic fiber f we have $\mathrm{Num}(S) = \mathbb{Z} \oplus \mathbb{Z}$, generated by the classes of C_0 and f.

Jumping sets of an ample line bundle were introduced in [12]. For the convenience of the reader their definition and basic properties are listed below.

Definition 2.1 ([12], Section 1)**.** Let L be an ample line bundle on X, spanned by a subspace $V \subseteq H^0(X, L)$. The set

$$\mathcal{J}_i(X, V) = \{x \in X | \mathrm{rk}(d\varphi_V)_x \leq n - i\}$$

is called the $i - th$ *jumping set* of (X, V). When $V = H^0(L)$ we set $\mathcal{J}_i(X, L) = \mathcal{J}_i(X, V)$.

Note that $\mathcal{J}_i(X, V)$ has a natural scheme-theoretic structure. The following chain of inclusions is immediate:

$$(2) \qquad \mathcal{J}_1(X, V) \supseteq \mathcal{J}_2(X, V) \supseteq \cdots \supseteq \mathcal{J}_n(X, V).$$

Observe that, according to the definition, $\mathcal{J}_1(X, V)$ is the locus where φ_V ramifies, while, as mentioned in the introduction,

$$\mathcal{J}_n(X, V) := \{x \in X \mid |V - x| = |V - 2x|\}.$$

Lemma 2.2. *Let X be a smooth variety of dimension n and let L be an ample and spanned line bundle on X. Suppose $\varphi_L : X \to Y$ is an $m : 1$ cover of a possibly singular n-dimensional variety Y. Let Δ be the branch locus of φ_L. Then*

$$\varphi_L(\mathcal{J}_2(X, L)) \subseteq (Sing(Y) \cup Sing(\Delta))$$

Proof. Let $x \in \mathcal{J}_2(X, L) \subseteq \mathcal{J}_1(X, L)$ and assume, by contradiction, that $y = \varphi_L(x) \notin \mathrm{Sing}(Y) \cup \mathrm{Sing}(\Delta)$. The smoothness of Y and Δ at y allows us to choose local coordinates $(y_1, \ldots, y_n)$ on Y centered at y and $(x_1, \ldots, x_n)$ on X centered at x such that in suitable neighborhoods of x and y the map φ_L can be written as:

$$y_1 = x_1^k$$
$$y_2 = x_2$$
$$(3) \qquad \ldots$$
$$y_n = x_n$$

where $2 \leq k \leq m$ and where $y_1 = 0$ is the local equation of Δ. This shows that $\mathrm{rk}\,(d\varphi_L)_x = n - 1$, and thus $x \notin \mathcal{J}_2(X, L)$, contradiction. $\square$ $\square$

For the convenience of the reader we also recall the statement of the so-called *Bertini's second theorem* and one of its corollaries. A nice historical account of the theorem can be found in Kleiman's article [11]. We sketch the proof of the corollary as we do not know of a literature reference for it.

THEOREM 2.3 (Bertini). *Let X be a complex projective variety. Let Σ be a linear system on X without fixed components, such that every $D \in \Sigma$ is reducible. Then Σ is composed with a pencil Λ.*

Corollary 2.4. *In the hypothesis of Theorem 2.3, if the base locus $\mathrm{Bs}\,\Sigma$ is nonempty and finite, then $\dim X = 2$ and the pencil Λ is rational.*

Proof. Let $p : X -- > Y \subseteq \mathbb{P}^N$ be the rational map given by Σ. The fact that Σ is composed with a pencil means that p factors through a curve C, say of genus g, i.e., $p = \sigma \circ \tau$, where $\tau : X -- > C$ has connected fibres, and $\sigma : C \to Y$ is a finite morphism. Any two general fibers of τ meet exclusively along the base locus of Σ. Because two general elements of a linear system meet in codimension one and because the base locus of Σ is finite, we conclude that $\dim X = 2$. Resolving the indeterminacies of the map p will produce at least one exceptional curve E that dominates C. Thus $g = 0$. $\qquad\square\qquad\qquad\square$

2.1. **General Results on Bad Points.** A reinterpretation of the above second Bertini theorem in our context, sheds light on the bad locus of an ample and spanned line bundle.

THEOREM 2.5. *Let L be an ample line bundle on a smooth complex projective variety X of dimension $n \geq 2$, spanned by a vector subspace $V \subseteq H^0(X, L)$. Let $x \in \mathcal{B}(X, V)$. Then*

    i) *$n = 2$;*
    ii) *There is an ample line bundle A on X with $h^0(A) \geq 2$ such that every $D \in |V - x|$ is of the form $D = A_1 + A_2 + \ldots A_r$, $r \geq 2$, $A_i \in \Lambda$, where $\Lambda \subseteq |A|$ is a linear pencil and thus $L \sim rA$;*
    iii) *$|V - x| = |V - rx|$ where r is as in ii);*
    iv) *$\mathcal{B}(X, V) \subseteq \mathcal{J}_n(X, V)$;*
    v) *$\varphi_V^{-1}(\varphi_V(x)) \subseteq \mathcal{B}(X, V)$.*

Proof. First of all note that because L is ample and spanned by V, the base locus of $|V - y|$ is a finite set for all $y \in X$, and in particular $|V - y|$ has no fixed component. As x is a bad point, Corollary 2.4 can be applied to the linear system $\Sigma = |V - x|$. Thus $n = 2$ and $|V - x|$ is composed with a rational pencil. This means that every $D \in |V - x|$ has the form $D = A_1 + \cdots + A_r$, $r \geq 2$ where $A_i \in \Lambda$, where Λ is a linear pencil contained in the complete linear system associated with

a line bundle A with $h^0(A) \geq 2$. Thus $L \sim rA$. Moreover $x \in A_i$ for some $i = 1, \ldots, r$. Because x belongs to all $D \in |V - x|$, necessarily x must belong to infinitely many elements $A_i \in \Lambda$, and therefore to all of them and thus x is a point of multiplicity $\geq r$ for every $D \in |V - x|$. Thus $|V - x| = |V - rx|$. In particular, since $r \geq 2$, this shows that $x \in \mathcal{J}_n(X, V)$ and thus $\mathcal{B}(X, V) \subseteq \mathcal{J}_n(X, V)$. Finally, notice that

$$(4) \qquad \mathrm{Bs}|V - x| = \{y_1, y_2, \ldots, y_s\} = \varphi_V^{-1}(\varphi_V(x)),$$

(where $y_1 = x$). In other words $|V - x| = |V - y_2| = \cdots = |V - y_s|$. In particular this shows that if $x \in \mathcal{B}(X, V)$, then every y_i is also in $\mathcal{B}(X, V)$, i. e.,

$$\mathrm{Bs}|V - x| = \varphi_V^{-1}(\varphi_V(x)) \subseteq \mathcal{B}(X, V). \qquad \square$$

$\square$

Corollary 2.6. *Let L be an ample line bundle on a smooth complex projective variety X of dimension $n = 2$, spanned by a vector subspace $V \subseteq H^0(X, L)$. Then*

  i) $\mathcal{B}(X, V)$ is a finite set;
  ii) $\mathcal{B}(X, L) = \emptyset$ if L is very ample.

Proof. Statement *i)* follows from the fact that $\dim(\mathcal{J}_i(X, V)) \leq n - i$, see [12], Theorem 1.2. Statement *ii)* is immediate from the above Theorem 2.5 and (2). $\square$ $\square$

The distinction between $\mathcal{B}(X, V)$ and $\mathcal{B}(X, L)$ for $V \subset H^0(X, L)$, is subtler than one might expect. Clearly $\mathcal{B}(X, L) \subseteq \mathcal{B}(X, V)$ for all nonempty subspaces V. The opposite inclusion is in general not true, as the following example shows.

EXAMPLE 2.7. Consider the pair $(S, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r))$ $(r \geq 2)$ and let $\Sigma \subset \mathbb{P}^N$ $(N = \binom{r+2}{2} - 1)$ be the image of S in the embedding φ_L. For any point $x \in S$ let $x' = \varphi_L(x)$ and let $\mathrm{Osc}_{x'}^{r-1}(\Sigma)$ be the $(r - 1)$-th osculating projective space to Σ at x'. Note that $\dim(\mathrm{Osc}_{x'}^{r-1}(\Sigma)) = \binom{r+1}{2} - 1$, as L is even r-jet ample, see [4]. Now let P be a general hyperplane of $\mathrm{Osc}_{x'}^{r-1}(\Sigma)$, so that P does not intersect Σ. Note that $N = \dim(P) + r + 2$. So we can consider the projection $\pi_P : \mathbb{P}^N - - \to \mathbb{P}^{r+1}$ from P to a $\mathbb{P}^{r+1}$ skew with P. Because P does not intersect Σ, the restricted map $\pi_{P|\Sigma} : \Sigma \to \mathbb{P}^{r+1}$ is a morphism. This means that the vector subspace $V \subset H^0(S, L)$ corresponding to that $\mathbb{P}^{r+1}$ spans L. Note that every hyperplane of $\mathbb{P}^N$ containing P and x' also contains $\mathrm{Osc}_{x'}^{r-1}(\Sigma)$, hence it is osculating of order $r - 1$ to Σ at x' and therefore the section it cuts out on Σ has a singular point of multiplicity r at x'. But such sections correspond to elements of the

linear system $|V - x|$. So this proves that $|V - x| = |V - rx|$. On the other hand the hyperplane sections of Σ are isomorphic to plane curves of degree r. Hence imposing a singular point of multiplicity r at x implies reducibility in r irreducible components passing through x. Therefore $x \in \mathcal{B}(S, V)$. Indeed $\mathcal{B}(S, V) = \mathcal{J}_2(X, V) = \{x\}$. Note however that $\mathcal{B}(S, L) = \emptyset$, since L is very ample. Moreover this shows that every $x \in S$ can become a bad point for a suitable $(r + 2)$-dimensional vector subspace $V \subset H^0(S, L)$ spanning L.

The following results establish connections between bad points and Seshadri constants. For background material on the latter, see [15], Section 5.

Proposition 2.8. *Let L be an ample and spanned line bundle on an n-fold X, $n \geq 2$. If $\epsilon(L, x) < 2$, then $x \notin \mathcal{J}_n(X, L)$; in particular $x \notin \mathcal{B}(X, L)$.*

Proof. Since $\epsilon(L, x) < 2$ there is an irreducible curve $C \subset X$ through x such that $LC < 2 \operatorname{mult}_x(C)$. Because $\operatorname{Bs}|L - x|$ is a finite set, there is an element $D \in |L - x|$ not containing C. Thus

$$2 \operatorname{mult}_x(C) > DC \geq \operatorname{mult}_x(D)\operatorname{mult}_x(C),$$

which shows that x is a smooth point of D. Hence $x \notin \mathcal{J}_n(X, L)$.   □
□

Recall that if ℓ is a line of (X, L) then $\epsilon(L, x) = 1$ for every $x \in \ell$ and thus one gets the following:

Corollary 2.9. *Let L be an ample and spanned line bundle on an n-fold X, $n \geq 2$. If (X, L) is covered by lines then $\mathcal{J}_n(X, L) = \mathcal{B}(X, L) = \emptyset$. In particular this happens for scrolls over bases of any dimension $\leq n - 1$.*

Lemma 2.10. *Let L be an ample and spanned line bundle on a surface X, with $\mathcal{B}(X, L) \neq \emptyset$. Let $L \sim rA$ be as in Theorem 2.5 and consider the subset $\mathcal{J}_2(A) = \{x \in X \,|\, |A - x| = |A - 2x|\}$.*

  i) *If $x \in X \setminus \mathcal{J}_2(A)$ then*

$$r \leq \epsilon(L, x) \leq rA^2;$$

  ii) *If $A^2 = 1$ then $X \setminus \mathcal{J}_2(A) = \{x \in X \,|\, \epsilon(L, x) = r\}$.*

Proof. Recall from Theorem 2.5 *ii)* that $h^0(A) \geq 2$. Let $x \in X \setminus \mathcal{J}_2(A)$ and let C be any irreducible curve through x. As $x \notin \mathcal{J}_2(A)$, there exists an element $\hat{A} \in |A - x|$ which is smooth at x. Thus it must be $LC = rAC = r\hat{A}C \geq r\operatorname{mult}_x(C)$ and thus $\epsilon(L, x) \geq r$. To see the right hand side inequality, choose an element $C \in |A - x|$. Let C_1 be

the irreducible component of C passing through x. Then $\frac{LC_1}{\mathrm{mult}_x(C_1)} \leq rAC = rA^2$ and thus $\epsilon(L, x) \leq rA^2$. This proves $i)$.

To see $ii)$ assume $A^2 = 1$ and $x \in X \setminus \mathcal{J}_2(A)$. Then $\epsilon(L, x) = r$ from $i)$. On the other hand, let $\epsilon(L, x) = r$ and let's choose $C \in |A - x|$. Then it must be $\mathrm{mult}_x(C) = 1$, which implies that $x \in X \setminus \mathcal{J}_2(A)$. $\square$

Lemma 2.11. *Let (S, L) be a polarized surface with L ample and spanned and $\mathcal{B}(S, L) \neq \emptyset$. Let A be as in Theorem 2.5, ii). Then $g(A) \geq 2$ unless $g(A) = 1$, $L = 2A$, and (S, A) is a Del Pezzo surface with $A^2 = 1$.*

Proof. If $g(A) = 0$, then A is very ample (see for example [8], (12.1), (5.1)) and therefore L is very ample which contradicts $\mathcal{B}(S, L) \neq \emptyset$. Let $g(A) = 1$. As scrolls have empty bad locus, see Corollary 2.9, (S, A) must be a Del Pezzo surface, according to [8], (12.3). If $A^2 \geq 3$ then A is very ample, contradiction. Therefore $A^2 = 1, 2$. If $A^2 = 2$, then $2A$ is very ample and therefore L is very ample, contradiction. If $A^2 = 1$, then $3A$ is very ample, therefore it must be $L = 2A$, i.e. (S, L) as in the statement. $\square$ $\square$

Lemma 2.12. *Let (S, L) be a polarized surface with L ample and spanned and $\mathcal{B}(S, L) \neq \emptyset$. Let A and r be as in Theorem 2.5, ii). Then*

   i) $h^0(L) \geq r + 2$, *in particular $h^0(L) \geq 4$;*
   ii) *Equality holds in i) if and only if $h^0(A) = 2$;*
   iii) *If equality holds in i) then φ_L is not a birational morphism.*

Proof. Recall that for two effective line bundles M and N on a smooth variety it is $h^0(M + N) \geq h^0(M) + h^0(N) - 1$, see [3], Lemma 1.1.6. Theorem 2.5 gives $L \sim rA$, and thus one can use the last inequality inductively to get $h^0((r - 1)A) \geq r$, as $h^0(A) \geq 2$. Now consider the sequence

$$(5) \qquad\qquad 0 \rightarrow (r - 1)A \rightarrow L \rightarrow L_{|A} \rightarrow 0.$$

Because L is ample and spanned, it must be

$$(6) \qquad\qquad k = \dim\left(Im(H^0(L) \rightarrow H^0(L_{|A}))\right) \geq 2.$$

Then (5) combined with (6) gives $h^0(L) = k + h^0((r - 1)A) \geq r + 2$.

Assume equality holds in $i)$. Then necessarily equality holds in (6) and $h^0(A) = 2$. Now assume $h^0(A) = 2$ and let $x \in \mathcal{B}(S, L)$. Then $|L - x|$ is composed with the pencil $|A|$, according to Theorem 2.5. Therefore $h^0(L - x) = \dim\left(Sym^r(\mathbb{C}^2)\right) = r + 1$. Because L is spanned, it is $h^0(L) = h^0(L - x) + 1 = r + 2$.

If equality holds in $i)$ then $h^0(A) = 2$ as above. Therefore $\varphi_{L_{|A}}$: $A \to \mathbb{P}^1$. As $g(A) \neq 0$, see Lemma 2.11, $\varphi_{L_{|A}}$ cannot be birational. Because $|A|$ is a pencil then φ_L is not birational. $\qquad\square \qquad\qquad\square$

3. Bad Points Exist

As already observed, the bad locus of an ample and spanned line bundle is a finite set and it is empty when $n \geq 3$. It is natural to look for an upper bound for its cardinality. Combining Theorem 2.5 with the inequality $\mathrm{Card}(\mathcal{J}_2(X,L)) \leq c_2(J_1L)$ [12], (2.6.1), where J_1L denotes the first jet bundle of L, we immediately get the upper bound $\mathrm{Card}(\mathcal{B}(X,L)) \leq c_2(J_1L)$. Note however that this bound is very unsatisfactory as easy examples show. Actually for $(X,L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ we have $\mathcal{B}(X,L) = \mathcal{J}_2(X,L) = \emptyset$, while $c_2(J_1L) = 3$. Similarly, let (X,L) be a surface scroll over a smooth curve. Then $\mathcal{B}(X,L) = \mathcal{J}_2(X,L) = \emptyset$, according to Corollary 2.9, while $c_2(J_1L) = L^2$. The many examples described in this section and the classification results contained in Section 4, suggest a better bound of the form $\mathrm{Card}(\mathcal{B}(X,L)) \leq \frac{L^2}{4}$. In most situations $\mathcal{B}(X,L)$ is actually empty, but this is not always the case, as the following example inspired by [2], p. 118-120 shows.

EXAMPLE 3.1. Let $\mathbb{F}_e$ be the Segre-Hirzebruch surface of invariant $e \geq 2$. For any integer $a \geq 2$ consider the line bundle $B = \mathcal{O}_{\mathbb{F}_e}(aC_0 + (ae-1)f)$. Note that the linear system $|(ae-1)C_0 + e(ae-1)f|$ contains a smooth irreducible curve, say R [9], Corollary 2.18 (b), p. 380. Let $\varphi : Y \to \mathbb{F}_e$ be a cyclic cover of degree e, branched along $C_0 + R \in |eB|$. Note that Y is a smooth surface, since C_0 and R are smooth and $C_0R = 0$. Let $E := \varphi^{-1}(C_0)$. Since C_0 is a component of the branch divisor we have $\varphi^*C_0 = eE$ and then we see from

$$e^2E^2 = (\varphi^*C_0)^2 = eC_0^2 = -e^2$$

that E is a (-1)-curve. Let $\sigma : Y \to X$ be its contraction. Then X is a smooth surface; call $x_0 \in X$ the point $\sigma(E)$. Recall that $\mathbb{F}_e$ is the desingularization $\nu : \mathbb{F}_e \to \Gamma_e$ of the cone $\Gamma_e \subset \mathbb{P}^{e+1}$ over the rational normal curve of degree e and that $C_0 + ef = \nu^*\mathcal{O}_{\Gamma_e}(1)$. Then we get the following commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \sigma\ } & X \\
\varphi \downarrow & & \downarrow \pi \\
\mathbb{F}_e & \xrightarrow[\ \nu\ ]{} & \Gamma_e
\end{array}
$$

where π exhibits X as a cyclic cover of degree e of the cone Γ_e branched at the vertex $v = \nu(C_0)$ and along the transverse intersection with a hypersurface of degree $ae - 1$. Note also that $\pi^{-1}(v) = \{x_0\}$.

Now, for any integer b, with $2 \le b \le a$, let $L_b := \pi^* \mathcal{O}_{\Gamma_e}(b - 1)$. Then L_b is an ample and spanned line bundle on X. For simplicity set $\widetilde{L}_b = \sigma^* L_b$. Then, due to the commutativity of the diagram above, we get

$$(7) \qquad \widetilde{L}_b = \varphi^*(\nu^* \mathcal{O}_{\Gamma_e}(b - 1)) = \varphi^* \mathcal{O}_{\mathbb{F}_e}((b - 1)(C_0 + ef)).$$

Recall that

$$\varphi_*(\mathcal{O}_Y) = \mathcal{O}_{\mathbb{F}_e} \oplus B^{-1} \oplus \cdots \oplus B^{-(e-1)}$$

(e. g. see [1], Lemma 17.2, p. 43). Then projection formula combined with $(2.1.1)$, and the inequality $b \le a$ give

$$h^0(L_b) = h^0(\widetilde{L}_b)$$
$$= h^0(\varphi_* \widetilde{L}_b)$$
$$= h^0((b - 1)(C_0 + ef)) + h^0((b - 1 - a)C_0 + (e(b - a) + 1 - e)f) + \ldots$$
$$= h^0(\nu^* \mathcal{O}_{\Gamma_e}(b - 1))$$
$$= h^0(\mathcal{O}_{\Gamma_e}(b - 1)).$$

Thus

$$(8) \qquad\qquad |L_b| = \pi^* |\mathcal{O}_{\Gamma_e}(b - 1)|.$$

Proposition 3.2. *Let (X, L_b) be as in the current Example. Then $\mathcal{J}_2(X, L_b) = \{x_0\}$ for every b with $2 \le b \le a$, while*

$$\mathcal{B}(X, L_b) = \begin{cases} \emptyset & \text{if } b > 2, \\ \{x_0\} & \text{if } b = 2. \end{cases}$$

Proof. The above construction shows that the branch locus of π is smooth, and therefore Lemma 2.2 gives $\mathcal{B}(X, L_b) \subset \mathcal{J}_2(X, L_b) \subset \{x_0\}$. By (8), the linear system $|L_b - x_0|$ consists of the elements $\pi^* D$ where D is a curve cut out on Γ_e by a hypersurface of degree $b - 1$ passing through v. Therefore for every $b \ge 2$ all elements in $|L_b - x_0|$ are singular at x_0, i. e., $x_0 = \mathcal{J}_2(X, L_b)$.

Let $b > 2$. Then the general element D as above is irreducible. Moreover, if D is in general position with respect to $\nu(R)$, then $\pi^* D$ itself is irreducible. This shows that $\mathcal{B}(X, L_b) = \emptyset$. On the contrary if $b = 2$ then all elements D as above consist of e lines of the ruling of Γ_e and then $x_0 = \mathcal{B}(X, L_b)$. $\qquad\qquad \square \qquad\qquad\qquad \square$

Note that
$$L_b^2 = e(b-1)^2(C_0 + ef)^2 = e^2(b-1)^2,$$
by (7). In particular $L_b^2 = 4$ if and only if $e = b = 2$. Moreover, apart from this case, $L_b^2 \geq 9$, with equality if and only if $e = 3, b = 2$.

Note also that if $e = a = b = 2$ then X is the Del Pezzo surface with $K_X^2 = 1$ and $L_b = -2K_X$ (e. g. see [3], Example 10.4.3, p. 269). In this case the equality $\mathcal{J}_2(X, L_b) = \{x_0\}$ was shown in [12], p. 206.

Remark 3.3. In the above case, $b = e = a = 2$, Lemma 2.10 gives
$$\{x \in X \,|\, \epsilon(L_b, x) = 2\} = X \setminus \mathcal{J}_2(-K_X),$$
where $\mathcal{J}_2(-K_X)$ is the set of double points of the singular elliptic curves, all irreducible, of the pencil $|-K_X|$. Notice that all elements of the anti-canonical system are smooth at the base point x_0, as $K_X^2 = 1$. This implies that $\{x_0\} = \mathcal{B}(X, L_b) \subset X \setminus \mathcal{J}_2(-K_X)$. Moreover
$$\mathcal{J}_2(-K_X) = \{x \in X \,|\, \epsilon(L_b, x) = 1\}.$$
Let $x \in \mathcal{J}_2(-K_X)$ and $C \in |-K_X - x|$. As C is an irreducible elliptic curve it is $\mathrm{mult}_x(C) = 2$ and thus $\epsilon(L_b, x) \leq 1$. On the other hand because L is spanned it is $\epsilon(L_b, x) \geq 1$, see [15].

If $e = b = 2, a = 3$, then X is a minimal surface of general type with $K_X^2 = 1$, $p_g(X) = 2$, studied by Horikawa, [10], and $L_b = 2K_X$. On the other hand $\mathrm{kod}(X) = 2$ except for the Del Pezzo surface just mentioned. In fact we have

Proposition 3.4. *Let X, $a \geq 2, e \geq 2$ be as in the current example. Then X is a minimal surface of general type unless $e = a = 2$.*

Proof. Recalling the expression of B, it is (e. g., see [1], Lemma 17.1, p. 42) that
$$K_Y = \varphi^*(K_{\mathbb{F}_e} + (e-1)B) = \varphi^*\mathcal{O}_{\mathbb{F}_e}(cC_0 + (ce-1)f)$$
where $c = ae - (a+2)$. On the other hand $K_Y = \sigma^*K_X + E$ and $\varphi^*C_0 = eE$. We thus get
$$(9) \qquad e\sigma^*K_X = eK_Y - \varphi^*C_0 = \varphi^*\mathcal{O}_{\mathbb{F}_e}(\lambda(C_0 + ef)),$$
where $\lambda = ce - 1$. Note that if $e \geq 3$ then $\lambda > 0$, while, if $e = 2$ then $\lambda = 2a - 5 > 0$ unless $a = 2$. So, except for the case $e = a = 2$, (9) shows that σ^*K_X is nef and big, so being $\mathcal{O}_{\mathbb{F}_e}(C_0 + ef)$. This in turn implies that K_X itself is nef and big. The nefness shows that X is minimal with $\mathrm{kod}(X) \geq 0$ and then the bigness gives $\mathrm{kod}(X) = 2$.   □   □

By the technique of bidouble covers we can construct another example of some interest in itself.

EXAMPLE 3.5. Let $\Gamma \subset \mathbb{P}^3$ be the quadric cone, let $\nu : \mathbb{F}_2 \to \Gamma$ be the desingularization, and let $v = \nu(C_0)$ be the vertex of Γ.

For any integer $\alpha \geq 1$ consider the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{F}_2}(\alpha(C_0 + 2f)$. The linear system $|2\mathcal{L}|$ contains a smooth divisor Δ_1. Let $\psi_1 : Z \to \mathbb{F}_2$ be the double cover branched along Δ_1. Then Z is a smooth surface; moreover, since $\Delta_1 C_0 = 0$ we have that $\psi_1^* C_0 = C_1 + C_2$, where C_1, C_2 are two smooth non-intersecting curves, both isomorphic to C_0 and exchanged by the involution defined by ψ_1. Thus the equality

$$2C_1^2 = 2C_2^2 = C_1^2 + C_2^2 = (C_1 + C_2)^2 = 2C_0^2 = -4$$

shows that both are (-2)-curves on Z. For any integer $\beta \geq 1$ consider on $\mathbb{F}_2$ the linear system $|(2\beta - 1)(C_0 + 2f)|$. We can find in it a smooth curve D, which is transverse to Δ_1 (to see this recall that $\mathcal{O}_{\mathbb{F}_2}((2\beta - 1)(C_0 + 2f)) = \nu^* \mathcal{O}_\Gamma(2\beta - 1))$. Thus $R = \psi_1^* D$ is a smooth curve on Z. Note that $DC_0 = 0$, hence $RC_1 = RC_2 = 0$. Moreover we have that $\mathcal{O}_{\mathbb{F}_2}(C_0 + D) \in 2\text{Pic}(\mathbb{F}_2)$. So if we consider the line bundle on Z given by $B = \psi_1^* \mathcal{O}_{\mathbb{F}_2}(\beta C_0 + (2\beta - 1)f)$, we see that $|2B|$ contains the smooth divisor $\Delta_2 := \psi_1^* C_0 + \psi_1^* D = C_1 + C_2 + R$. Let $\psi_2 : Y \to Z$ be the double cover branched along Δ_2. Then Y is a smooth surface; moreover since C_i is in the branch divisor for $i = 1, 2$, we have that $\psi_2^* C_i = 2E_i$, where E_i is a smooth rational curve on Y. Thus $4E_i^2 = 2(C_i)^2 = -4$, which means that E_i is a (-1)-curve on Y. Note also that $E_1 E_2 = 0$ since $C_1 C_2 = 0$. Let $\sigma : Y \to X$ be the birational morphism contracting E_1 and E_2 and let $x_i = \sigma(E_i)$. Call $\pi : X \to \Gamma$ the finite morphism of degree 4 induced by the bidouble cover $\psi_1 \circ \psi_2 : Y \to \mathbb{F}_2$ via σ and ν. The following commutative diagram is obtained:

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \sigma\ } & X \\
{\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\pi} \\
Z & & \\
{\scriptstyle\varphi_1}\downarrow & & \\
\mathbb{F}_2 & \xrightarrow{\ \nu\ } & \Gamma
\end{array}
$$

Note that $\pi(x_1) = \pi(x_2) = v$ and v is a branch point of π, by construction. Now set $L_c = \pi^* \mathcal{O}_\Gamma(c - 1)$, for any integer $c \geq 2$. Then L_c is an ample and spanned line bundle on X. For shortness set $\widetilde{L}_c = \sigma^* L_c$. Then, due to the commutativity of the diagram above, it follows

$$(10) \qquad \widetilde{L}_c = \psi_2^*(\psi_1^* \mathcal{O}_{\mathbb{F}_2}((c-1)(C_0 + 2f))).$$

Thus, by arguing as in Example 3.1, it is easy to see that if $c \leq \min\{\alpha, \beta\}$ then

$$(11) \qquad |L_c| = \pi^* |\mathcal{O}_\Gamma(c-1)|.$$

This is the key to prove, in the same way as we did in Proposition 3.2, that $\mathcal{J}_2(X, L_c) = \{x_1, x_2\}$ for every c with $2 \leq c \leq \min\{\alpha, \beta\}$, while

$$\mathcal{B}(X, L_c) = \begin{cases} \emptyset & \text{if } c > 2, \\ \{x_1, x_2\} & \text{if } c = 2. \end{cases}$$

Finally note that $L_c^2 = \widetilde{L}_c^2 = 4(c-1)^2(C_0 + 2f)^2 = 8(c-1)^2$. In particular $L_c^2 \geq 8$ with equality only if $c = 2$. In this case we observe that every element of $|L_c - x_i| = |L_c - x_1 - x_2|$ splits in two components, which are the images via σ of two fibres of $Y \to \mathbb{P}^1$ belonging to the pencil $|\psi_2^*(\psi_1^* f)|$.

4. Polarized Surfaces of low degree with non trivial bad loci

The examples studied in the previous section and Lemma 2.2 suggest that non empty bad loci commonly occur on surfaces that are covers of cones. The fact that L is divisible in $\mathrm{Pic}(S)$, as shown in Theorem 2.5, imposes simple stringent conditions on the possible degree of a polarized surface admitting a non trivial bad locus. In particular it immediately follows from Theorem 2.5 that $L^2 = 4, 8, 9$ or $L^2 \geq 12$. In what follows, polarized surfaces with non trivial bad loci of degree up to 9 or equal to the square of a prime p are completely classified. Indeed they turn out to be finite covers of cones, as anticipated in the examples of the previous section.

Proposition 4.1. *Let (S, L) be a polarized surface with L ample and spanned. Assume $\mathcal{B}(S, L) \neq \emptyset$. Let $L^2 = p^2$ where p is prime. Then*

  i) *$\mathcal{B}(S, L) = \{x\}$;*
  ii) *$\varphi_L : S \to Y_p$ expresses S as a p-uple cover of a cone $Y_p \subset \mathbb{P}^{p+1}$ over a rational normal curve of degree p, whose branch locus contains the vertex v;*
  iii) *$\varphi_L(x) = v$;*
  iv) *In particular, if $p = 2$ then $L = 2A$ where (S, A) is a Del Pezzo surface of degree one.*

Proof. Let $x \in \mathcal{B}(S, L)$. Theorem 2.5 gives $L = pA$ with $A^2 = 1$. This implies that every pair of divisors in the pencil $\Lambda \subseteq |A|$ meet only at x. As $|L - x|$ does not have a fixed component, it is set theoretically $\mathrm{Bs} |L - x| = \mathrm{Bs}\, \Lambda = \{x\}$. Bertini's theorem then gives that the generic $A_i \in \Lambda$ is everywhere smooth. Let us consider a generic smooth $A \in \Lambda$. If $g(A) = 1$ Lemma 2.11 implies that $p = 2$ and $A = -K_S$. It is well known and easily seen that in this case φ_{-2K_S} expresses S as a double cover of a quadric cone in $\mathbb{P}^3$, see for example [3], Ex. 10.4.3, p. 269. Therefore $g(A) \geq 2$, and as $A^2 = 1$ the sequence $0 \to \mathcal{O}_S \to A \to A_{|A} \to 0$ gives $h^0(A) = 2$. Lemma 2.12 then gives $h^0(L) = p + 2$, and φ_L not birational. In particular $\deg(\varphi_L(S)) \geq 2$. From

$$(12) \qquad\qquad p^2 = \deg(\varphi_L) \deg(\varphi_L(S))$$

we see that φ_L must be a p-uple cover of a surface $Y_p \subset \mathbb{P}^{p+1}$ of degree p. As Y_p is covered by lines $\varphi_L(A_i)$ for $A_i \in |A|$, all of which meet at one point, $\varphi_L(x)$, Y_p must be a cone with vertex $v = \varphi_L(x)$, over a rational normal curve of degree p. Because v is the only point on Y_p through which there are infinitely many lines, $\mathcal{B}(S, L) = x$. When $p = 2$ it follows from [7], Section 4 that (S, L) is as in *iv).*   □   □

Proposition 4.2. *Let (S, L) be a smooth surface polarized with an ample and spanned line bundle with $L^2 = 8$. Assume that $\mathcal{B}(S, L) \neq \emptyset$. Then*

 i) *$\mathcal{B}(S, L) = \{x_1, x_2\}$ with $x_1 \neq x_2$;*
 ii) *$\varphi_L : S \to Y_2$ expresses S as a quadruple cover of a quadric cone $Y_2 \subset \mathbb{P}^3$ with vertex v;*
 iii) *$\varphi_L(x_1) = \varphi_L(x_2) = v$.*

Proof. Let $x \in \mathcal{B}(S, L)$ and let C be a generic element in $|L - x|$. Note that $\mathrm{mult}_x(C) \geq 3$ would imply $L^2 \geq 9$, and thus it is $\mathrm{mult}_x(C) = 2$. If there exists $y \in \mathrm{Bs} |L - x|$, $y \neq x$, Theorem 2.5 implies that y is also a bad point for L and thus $\mathrm{mult}_y(C) \geq 2$. These facts, combined with the second Bertini Theorem, imply that the only two possible configurations for $\mathrm{Bs} |L - x|$ are as follows:

 a) $\mathrm{Bs} |L - x| = \{x\}$ with the intersection index at x $(C_1, C_2)_x = L^2 = 8$ for two general elements of $|L - x|$.
 b) $\mathrm{Bs} |L-x| = \{x_1, x_2\}$ with $x_1 \neq x_2$, and $(C_1, C_2)_{x_1} = (C_1, C_2)_{x_2} = 4$ for all $C_i \in |L - x|$.

Claim Case a) does not happen.

In case a), the general $C \in |L - x|$ would be smooth away from x and two general such curves would intersect only at x. This fact, combined with the two Bertini theorems, forces the general C to be reducible at

x with $C = A + B$, A and B both smooth at x and linearly equivalent, $A^2 = AB = B^2 = 2$. This implies that the tangent lines to A and B at x must be the same and that this line does not change when we vary C generically in $|L - x|$. Let now $C_i \in |L - x|$, $i = 1, 2$ be two general curves, and consider the pencil $\mathcal{P} = < C_1, C_2 >$. Choosing suitable local coordinates z_1, z_2 centered at x, such that the common tangent at x has equation $z_1 = 0$, the local equations of the C_i are $z_1^2 + F_i(z_1, z_2) = 0$ where the F_i's are polynomials of degree at least 3. Therefore $\mathcal{P}$ contains the curve $C_1 - C_2$ which has at least a triple point at x. As $L^2 = 8$, there can be only one such curve in $|L - x|$. This means that $|L - x| = \mathcal{P}$. Indeed, if $\mathcal{P} \subset |L - x|$, we could find a general $C_3 \in |L - x|, C_3 \notin \mathcal{P}$. Now with C_1 and C_3 we could repeat the above pencil argument and produce a second curve with a triple point, different from the first one, which is a contradiction. Therefore $h^0(L - x) = 2$ and, because L is ample and spanned, $h^0(L) = 3$, but this contradicts Lemma 2.12, $i)$. This concludes the proof of the Claim.

Let now $|L - x|$ be as in b). Again Bertini's theorems give $L = 2A$ where $A \in \Lambda$, a pencil in $|A|$, and that the generic $A \in \Lambda$ is smooth everywhere and goes through both x_i's. Notice that $\{x_1, x_2\} \subseteq \mathcal{B}(S, L) \subseteq \mathcal{J}_2(S, L) \subseteq \mathcal{J}_1(S, L)$ and thus the x_i's are both ramification points for the finite map φ_L. Because $\varphi_L(x_1) = \varphi_L(x_2)$, it follows that $\deg(\varphi_L) \geq 4$. As $L^2 = 8$, $\deg(\varphi_L(S)) \leq 2$, and thus $h^0(L) \leq 4$. Therefore Lemma 2.12 gives $h^0(L) = 4, h^0(A) = 2$ and φ_L not birational. Thus the sequence $0 \to A \to L \to L_{|A} \to 0$ gives $\dim(Im(H^0(L) \to H^0(L_{|A}))) = 2$. Therefore $\varphi_{L_{|A}}(A)$ expresses A as a $4 : 1$ cover of a line. As the pencil $|A|$ sweeps out the whole S, it must be $\deg(\varphi_L) \geq 4$ and thus $\deg(\varphi_L(S)) = 2$. Let $\mathbb{Q} = \varphi_L(S)$. As $\mathbb{Q}$ is swept by a pencil of concurrent lines, images of the pencil of curves $|A|$, $\mathbb{Q}$ must be a quadric cone with vertex $v = \varphi_L(x_i)$. Because v is the only point on $\mathbb{Q}$ through which there are infinitely many lines, $\mathcal{B}(S, L) = \{x_1, x_2\}$.   □   □

5. The rude locus

The following set of examples is aimed at showing how the behavior of the rude locus of an ample and free linear system covers a very wide spectrum of possibilities. As mentioned in the introduction, $\mathcal{R}(X, V)$ can be empty, a finite set, a divisor minus a finite set, a divisor, a dense Zariski open subset, the union of a dense Zariski open subset and a finite set or the whole variety.

Let (X, L) be a polarized n-fold with $V \subseteq H^0(L)$ a subspace that spans L. In all the following examples we will denote with s the projective dimension of the linear system $|V|$. Notice that our assumptions on L and V imply that $s = \dim(|V|) \geq n$. As customary we set the dimension of the empty set equal to -1.

Lemma 5.1. *Let (X, L) be a polarized $n - fold$, and $V \subseteq H^0(L)$ be a subspace that spans L. Let $s = \dim(|V|)$, let $x \in X$ and let φ_V be the map associated with $|V|$. Then*

i) *$\dim(|V - 2x|) = s - 1 - \mathrm{rk}(d\varphi_V)_x$;*

ii) *$\dim(|V - 2x|) \geq s - (n+1)$ with equality holding if and only if $x \in X \setminus \mathcal{J}_1(X, V)$;*

iii) *If $\dim(|V|) = n$ then $\mathcal{R}(X, V) \subseteq \mathcal{J}_1(X, V)$ and $|V - 2x| \neq \emptyset$ for all $x \in \mathcal{J}_1(X, V)$.*

Proof. Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$ and consider the homomorphism

$$j_{1,x} : V \to L \otimes \mathcal{O}_X/\mathfrak{m}_x^2$$

sending every section $s \in V$ to its first jet at x, i.e. $j_{1,x}(s) = (s(x), ds(x))$ in a local chart around x. Because V spans L at x it is $\mathrm{rk}(j_{1,x}) = 1 + \mathrm{rk}(d\varphi_V)_x$. Then *i)* follows noting that $|V - 2x| = \mathbb{P}(Ker(j_{1,x}))$. To see *ii)* notice that $\mathrm{rk}(d\varphi_V)_x \leq n$ with equality if and only if $x \in X \setminus \mathcal{J}_1(X, L)$. Assuming $s = n$, *ii)* gives *iii)*. □   □

EXAMPLE 5.2 (The Veronese surface)**.** Let $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ and let $V \subseteq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ be a subspace that spans L. Let $s = \dim(|V|)$. Then it is

$$\mathcal{R}(X, V) = \begin{cases} X & \text{if } s \geq 3, \\ \mathcal{J}_1(X, V) & \text{if } s = 2. \end{cases}$$

Assume first that $s \geq 3$. Because every singular conic is necessarily reducible, it is enough to show that for every $x \in X$, $|V - 2x|$ is not empty. This follows from the condition $s \geq 3$ and Lemma 5.1, *ii)*.

Now assume $s = 2$. Because of Lemma 5.1 we need to show that if $x \in \mathcal{J}_1(X, V)$ then $x \in \mathcal{R}(X, V)$. It is enough to show that $|V - 2x|$ is nonempty. This follows again from Lemma 5.1, *ii)*.

The fact that being singular and being reducible are equivalent for rational curves is responsible for the peculiarity of the two dimensional case among the polarized varieties $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$. The following example describes the general picture.

EXAMPLE 5.3. Let $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$, with $n \geq 2$ and let $V \subseteq H^0(X, L)$ be a general subspace, with $s = \dim(|V|) \geq n + 1$.

Let $\mathcal{W} \subset |L|$ be the subvariety parameterizing all reducible quadric hypersurfaces of $\mathbb{P}^n$.

Notice that $\mathcal{W}$ is isomorphic to the second symmetric power of the dual of $\mathbb{P}^n$, hence $\dim(\mathcal{W}) = 2n$. If $\mathcal{W} \cap |V| = \emptyset$, then $\mathcal{R}(X, V) = \emptyset$. Now suppose that $\mathcal{W}$ meets $|V|$, and inside $X \times (|V| \cap \mathcal{W})$ consider the following incidence variety:

$$\mathcal{I} := \{(x, D) | x \in \operatorname{Sing}(D)\}.$$

Let p and q be the morphisms induced on $\mathcal{I}$ by the projections of $X \times (|V| \cap \mathcal{W})$ onto the factors. Notice that

$$p^{-1}(x) = \{D \in |V| \cap \mathcal{W} \mid x \in \operatorname{Sing}(D)\} = |V - 2x| \cap \mathcal{W},$$

and therefore $x \in \mathcal{R}(X, V)$ if and only if

(13) $$p^{-1}(x) = |V - 2x|.$$

We claim that (13) cannot occur for $n \geq 3$, and for a general $x \in X$.

Since V is general, the codimension of $(|V| \cap \mathcal{W})$ in $|V|$ is the same as that of $\mathcal{W}$ in $|L|$, hence

$$\dim(|V| \cap \mathcal{W}) = s - \left(\binom{n+2}{2} - 1 - 2n\right) = s - \binom{n}{2}.$$

As the singular locus of an element $D \in |V|$ broken into two distinct hyperplanes is a linear space of codimension 2, the general fiber of q has dimension $n - 2$. Therefore $\dim(\mathcal{I}) = s - \binom{n}{2} + n - 2$ and then for the general $x \in X$ we have that $\dim(p^{-1}(x)) = s - \frac{1}{2}(n^2 - n + 4)$.

Therefore Lemma 5.1 gives for a general x:

(14)
$$\dim(|V - 2x|) - \dim(p^{-1}(x)) \geq s - n - 1 - (s - \frac{1}{2}(n^2 - n + 4))$$

$$= \frac{1}{2}(n-1)(n-2)$$

which proves the claim.

This discussion proves that $\mathcal{R}(X, V)$ is contained in (or is equal to) a closed Zariski subset of X. It is easy to see that $\mathcal{R}(X, L) = \emptyset$.

The following example shows that the inclusion given in Lemma 5.1 *iii)* can be proper. Notice again the striking difference between the behavior of $\mathcal{R}(X, V)$ in dimension two versus the higher dimensional cases.

EXAMPLE 5.4 (Del Pezzo manifolds of degree 2). Let (X, L) be a Del Pezzo manifold of dimension $n \geq 2$, i.e. $-K_X = (n-1)L$ with L ample, $L^n = 2$. Then L is spanned, $h^0(L) = n + 1$ and $\varphi_L : X \to \mathbb{P}^n$

is a double cover ramified over a smooth quartic hypersurface Δ. For $n = 2$, recall that a smooth plane quartic admits 28 double-tangent lines ℓ_k. Let $\Delta \cap \ell_k = \{y_1^k, y_2^k\}$ and consider the 56 points $x_j^k = \varphi_L^{-1}(y_j^k)$, $j = 1, 2$, $k = 1, \ldots, 28$. If V is a subspace that spans L, necessarily $|V| = |L|$ because $\dim(|L|) = n$. Then

$$\mathcal{R}(X, L) = \begin{cases} \emptyset & \text{if } n \geq 3, \\ \{x_j^k | k = 1, \ldots, 28, \ \ j = 1, 2\} & \text{if } n = 2. \end{cases}$$

Assume first that $n \geq 3$. Then $\mathrm{Pic}(X) \simeq \mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$ generated by L, see for example [14], Prop. 3.1. If $x \in \mathcal{R}(X, L)$ and $D \in |L - 2x|$ then it must be $D = A + B$ for two effective divisors A and B. Thus $A = aL$ and $B = bL$ where $a, b \geq 1$. This gives the following chain of equalities in $\mathrm{Pic}(X)$

$$L = D = A + B = (a + b)L$$

which is a contradiction as $a + b \geq 2$.

Let now $n = 2$. Lemma 5.1 gives $\mathcal{R}(X, L) \subseteq \mathcal{J}_1(X, L)$. For every point $x \in \mathcal{J}_1(X, L)$ the elements in $|L - 2x|$ are the preimages via φ_L of the lines in $\mathbb{P}^2$ which are tangent to Δ at $y = \varphi_L(x)$. For a generic x, such a tangent line is tangent to Δ only at y, and therefore the corresponding element in $|L - 2x|$ is an irreducible curve with arithmetic genus one, with one double point. However $|L - 2x_j^k|$ consists of a curve, $\varphi_L^{-1}(\ell_k)$, of arithmetic genus one with two double points x_1^k, x_2^k and therefore reducible. Thus $\mathcal{R}(X, L) = \{x_j^k | k = 1, \ldots, 28, \ \ j = 1, 2\}$. Notice that here the inclusion $\mathcal{R}(X, L) \subset \mathcal{J}_1(X, L)$ is strict.

EXAMPLE 5.5 (Del Pezzo manifolds of degree 1)**.** Let $(X, \mathcal{L})$ be a Del Pezzo manifold of degree one, i.e. $-K_X = (n-1)\mathcal{L}$, with $\mathcal{L}$ ample and $\mathcal{L}^n = 1$. Then $h^0(X, \mathcal{L}) = n$. Moreover $\mathcal{L}$ is not spanned and $\mathrm{Bs}|\mathcal{L}|$ consists of a single point x^*. Set $L = 2\mathcal{L}$. The line bundle L is ample and spanned, with $h^0(X, L) = \binom{n+1}{2} + 1$ and it defines a $(2 : 1)$ cover, $\varphi_L : X \to \Gamma$ of a cone over the 2-Veronese manifold of dimension $n - 1$, $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(2))$, where $\varphi_L(x^*) = v$, the vertex of the cone. For general results on Del Pezzo manifolds see [8], Chapter I.

Even though the line bundle $\mathcal{L}$ is not spanned, we set

$$\mathcal{J}_i(\mathcal{L}) = \{x \in X \setminus \{x^*\} | \mathrm{rk}\,(d\varphi_{\mathcal{L}})_x \leq n - i\}.$$

Remark 5.6. If $n = 2$, as $\mathrm{rk}(\varphi_L)_x \leq 1$, it is $\mathcal{J}_1(\mathcal{L}) = X \setminus x^*$. Moreover $\mathcal{J}_2(\mathcal{L})$ is given by the finite set of the singularities $x_1, \ldots, x_\delta$ of the singular curves in the pencil $|\mathcal{L}|$. It is well known that if X is general in the moduli space of Del Pezzo surfaces of degree 1 then $|\mathcal{L}|$ contains

exactly 12 singular curves, each of which has a single ordinary node, hence $\delta = 12$.

Lemma 5.7.
$$\mathcal{J}_{i+1}(\mathcal{L}) \subseteq \mathcal{J}_i(X, L) \subseteq \mathcal{J}_i(\mathcal{L}) \cup \{x^*\}.$$

Proof. First notice that $x^* \in \mathcal{J}_n(X, L) \subseteq \mathcal{J}_i(X, L)$ for all $1 \leq i \leq n$ because every $D \in |L - x^*|$ is of the form $D = \varphi_L^*(H)$ where H is a hyperplane section of Γ through v and thus $D \in |L - 2x^*|$.

Let now $x \neq x^*$. Let us fix a basis for $H^0(X, \mathcal{L})$, $\mathfrak{B} = \{s_0, \dots, s_{n-1}\}$. Let $\mathfrak{B}_1$ be the basis of $Sym^2(H^0(X, \mathcal{L}))$ constructed from $\mathfrak{B}$. We can choose an appropriate section h such that $\mathfrak{B}_1 \cup \{h\}$ is a basis for $H^0(X, L)$.

Assume $\mathrm{rk}\,(d\varphi_{\mathcal{L}})_x = t$. Then, after choosing local coordinates around x, say $x_1, \dots x_n$, and renumbering the sections in $\mathfrak{B}$ we can assume that in a neighborhood of x it is $s_0 = 1$, $s_i = x_i + \textit{higher order terms}$ for $i = 1, \dots, t$ and $s_j \in H^0(\mathcal{L} - 2x)$ for $j = t + 1, \dots, n - 1$. From our construction of the basis for $H^0(X, L)$ it follows that the only linearly independent rows in the matrix $(d\varphi_L)_x$ are given by $(d(s_0 s_i))_x$, for $i = 1 \dots t$, and possibly $(dh)_x$. Thus $\mathrm{rk}\,(d\varphi_L)_x = t$ or $t + 1$. The above argument shows that

$$\mathrm{rk}\,(d\varphi_L)_x \leq \mathrm{rk}\,(d\varphi_{\mathcal{L}})_x + 1$$

from which we deduce that $\mathcal{J}_{i+1}(\mathcal{L}) \subseteq \mathcal{J}_i(X, L)$.

Assume now that $\mathrm{rk}\,(d\varphi_L)_x = t$. Then after choosing appropriate local coordinates, $y_1, \dots, y_n$, around x, we can assume that there are $t + 1$ global sections $h_0, \dots, h_t \in \mathfrak{B}_1 \cup \{h\}$ that locally can be expressed as $h_0 = 1$, $h_k = y_k + \textit{higher order terms}$ for $k = 1, \dots, t$. If $h_k \neq h$ for all $k = 0, \dots, t$ then $h_k \in \mathfrak{B}_1$ and thus $h_k = s_{i_k} s_{j_k}$, with $s_{i_k}, s_{j_k} \in \mathfrak{B}$. The expansions of the $h_k's$ in local coordinates show that there are $t + 1$ sections in $\mathfrak{B}$, $s_{i_0}, \dots, s_{i_t}$ such that $h_0 = s_{i_0} s_{i_0}$ and $h_k = s_{i_0} s_{i_k}$. Locally it is $s_{i_0} = 1$, $s_{i_k} = y_k + \textit{higher order terms}$ for $k = 1, \dots, t$. Those sections form a non zero $t \times t$ minor for the matrix $(d\varphi_{\mathcal{L}})_x$. It follows that $\mathrm{rk}\,(d\varphi_{\mathcal{L}})_x = t$. If there is a k such that $h_k = h$, then, reasoning as above, $\mathrm{rk}\,(d\varphi_{\mathcal{L}})_x = t$ or $\mathrm{rk}\,(d\varphi_{\mathcal{L}})_x = t - 1$. The above argument shows that

$$\mathrm{rk}\,(d\varphi_L)_x - 1 \leq \mathrm{rk}\,(d\varphi_{\mathcal{L}})_x \leq \mathrm{rk}\,(d\varphi_L)_x$$

from which we deduce that $\mathcal{J}_i(X, L) \subseteq \mathcal{J}_i(\mathcal{L}) \cup \{x^*\}$. $\square$ $\square$

Lemma 5.8. *Let (X, L) be as above with $n = 2$. Then*
$$\mathcal{R}(X, L) = (X \setminus \mathcal{J}_1(X, L)) \cup \mathcal{J}_2(\mathcal{L}) \cup \{x^*\}$$

where $\mathcal{J}_2(\mathcal{L})$ is as in Remark 5.6.

Proof. In dimension 2, $\varphi_L : X \to \Gamma$ is a double cover of a quadric cone in $\mathbb{P}^3$, ramified along the vertex and Δ, the intersection of a smooth hypersurface of degree 3 with Γ. The jumping loci are worked out in [12] where it is shown that $\mathcal{J}_2(X, L) = \{x^*\}$ and $\mathcal{J}_1(X, L) = \mathfrak{R} \cup \{x^*\}$, where $\mathfrak{R}$ is the ramification divisor $\varphi_L^{-1}(\Delta)$. If $x \notin \mathcal{J}_1(X, L)$ then $\dim(|L - 2x|) = 0$ i.e. there is a unique singular divisor in $|L - x|$, which must be of the form $2D$, where D is the unique divisor in $|\mathcal{L} - x|$. Hence $X \setminus \mathcal{J}_1(X, L) \subset \mathcal{R}(X, L)$. We have already seen that $x^* \in \mathcal{B}(X, L) \subset \mathcal{R}(X, L)$. When $x^* \neq x \in \mathcal{J}_1(X, L)$ then $\dim(|L - 2x|) = 1$, i.e. there is a pencil of singular sections in $|L - x|$. A section $D \in |L - 2x|$ is given by $\varphi_L^* H$, where H is a hyperplane section of the cone Γ, tangent to $\varphi_L(\mathfrak{R}) = \Delta$ at $\varphi_L(x)$, i.e. given by an element of the pencil of planes containing the tangent line to Δ at $\varphi_L(x)$. The generic such H is irreducible, unless the tangent line ℓ_x to Δ at $\varphi_L(x)$ is a line of the cone through v, which corresponds to the unique element $T \in |\mathcal{L} - x|$. This happens only if $x \in \mathcal{J}_2(\mathcal{L})$. Notice that in this case every element of the pencil of singular sections is reducible as $T + \hat{T}$, with $\hat{T}$ varying in $|\mathcal{L}|$ and thus $\mathcal{J}_2(\mathcal{L}) \subseteq \mathcal{R}(X, L)$.   □   □

Lemma 5.9. *Let $n \geq 3$ and let $x \in \mathcal{R}(X, L)$, then*

$$\dim(|L - 2x|) = \begin{cases} 2n - 4 & \text{if } x \notin \mathcal{J}_n(\mathcal{L}) \cup \{x^*\}, \\ 2n - 3 & \text{if } x \in \mathcal{J}_n(\mathcal{L}), \\ 2n - 2 & \text{if } x = x^*. \end{cases}$$

Proof. Consider the following two families of divisors on X

$$\mathcal{F} = \{D_1 + D_2|\ D_i \in |\mathcal{L} - x|\ i = 1, 2\}$$

and

$$\mathcal{G} = \{D_1 + D_2|\ D_1 \in |\mathcal{L} - 2x|, D_2 \in |\mathcal{L}|\}.$$

There is a natural $2 : 1$ cover $|\mathcal{L} - x| \times |\mathcal{L} - x| \to \mathcal{F}$ and thus

(15) $$\dim(\mathcal{F}) = \begin{cases} 2n - 4 & \text{if } x \neq x^*, \\ 2n - 2 & \text{if } x = x^*. \end{cases}$$

Furthermore, $|\mathcal{L} - 2x| \times (|\mathcal{L}| \setminus |\mathcal{L} - x|) = |\mathcal{L} - 2x| \times \mathbb{C}^{n-1}$ is a dense Zariski open subset of $\mathcal{G}$. Thus

(16) $$\dim(\mathcal{G}) = (n - 1) + \dim(|\mathcal{L} - 2x|).$$

Moreover it is

(17) $$\dim(|\mathcal{L} - 2x|) = \begin{cases} n - 2 & \text{if } x \in \mathcal{J}_n(\mathcal{L}), \\ \leq n - 3 & \text{if } x \in (X \setminus \mathcal{J}_n(\mathcal{L})) \cup \{x^*\}. \end{cases}$$

Because $\mathrm{Pic}(X) = \mathbb{Z}$ generated by $\mathcal{L}$, a divisor $D \in |L - 2x|$ is reducible if and only if $D \in \mathcal{F} \cup \mathcal{G}$. If $x \in \mathcal{R}(X, L)$ then $\dim(|L - 2x|) = \max\{\dim(\mathcal{F}), \dim(\mathcal{G})\}$. Putting together (15), (16), and (17) one obtains the statement. $\qquad \square \qquad\qquad\qquad \square$

Proposition 5.10. *Let $(X, \mathcal{L})$ be a Del Pezzo manifold of dimension n with $\mathcal{L}^n = 1$. Let $L = 2\mathcal{L}$. Then*

$$\mathcal{R}(X, L) = \begin{cases} \emptyset & \text{for } n \geq 4, \\ X \setminus \mathcal{J}_1(X, L) & \text{for } n = 3, \\ (X \setminus \mathcal{J}_1(X, L)) \cup \mathcal{J}_2(\mathcal{L}) \cup \{x^*\} & \text{for } n = 2. \end{cases}$$

Proof. Lemma 5.8 gives the statement for $n = 2$. Combining Lemma 5.9, Lemma 5.8 and Lemma 5.1 *i)*, it follows that $\mathcal{R}(X, L) = \emptyset$ if $n \geq 5$ and $x^* \in \mathcal{R}(X, L)$ if and only if $n = 2$. If $n = 4$ Lemma 5.9 implies that if $x \in \mathcal{R}(X, L)$ then $x \in \mathcal{J}_4(\mathcal{L}) \cap (X \setminus \mathcal{J}_1(X, L))$ but this intersection is empty according to Lemma 5.7. If $n = 3$ and $x \in \mathcal{R}(X, L)$ then Lemma 5.9 gives

$$x \in [\mathcal{J}_3(\mathcal{L}) \cap (\mathcal{J}_1(X, L) \setminus \mathcal{J}_2(X, L))] \cup [(X \setminus \mathcal{J}_3(\mathcal{L})) \cap (X \setminus \mathcal{J}_1(X, L))].$$

Lemma 5.7 gives

$$\mathcal{J}_3(\mathcal{L}) \cap (\mathcal{J}_1(X, L) \setminus \mathcal{J}_2(X, L)) = \emptyset$$

and

$$(X \setminus \mathcal{J}_3(\mathcal{L})) \cap (X \setminus \mathcal{J}_1(X, L)) = X \setminus \mathcal{J}_1(X, L).$$

Thus $\mathcal{R}(X, L) \subseteq X \setminus \mathcal{J}_1(X, L)$. Following the proof of Lemma 5.9, and using the same notation introduced there, notice that when $n = 3$ it is $|\mathcal{L} - x| \simeq \mathbb{P}^1$ and thus $\mathcal{F} \simeq \mathbb{P}^2$. If $x \in X \setminus \mathcal{J}_1(X, L)$ then $\dim(|L - 2x|) = 2$ and thus $|L - 2x| = \mathcal{F}$ i.e. $x \in \mathcal{R}(X, L)$. $\qquad \square \qquad\qquad \square$

EXAMPLE 5.11 $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$**.** Let $X = \mathbb{Q}^n$ and $L = \mathcal{O}_{\mathbb{Q}^n}(1)$. Let $V \subset H^0(X, L)$ be a spanning subspace. Then either $|V| = |L|$ or $\mathrm{codim}_{|L|}(|V|) = 1$. In the first case if $n = 2$ it is immediate that $\mathcal{R}(X, L) = X$ while if $n \geq 3$ a generic hyperplane, tangent to X at one point, cuts on X an irreducible quadric cone and thus $\mathcal{R}(X, L) = \emptyset$. Let now V be a spanning subspace of codimension one. Let us first observe that $\varphi_V : X \to \mathbb{P}^n$ is a double cover, ramified along a smooth quadric hypersurface $\Delta = \mathbb{Q}^{n-1} \subset \mathbb{P}^n$. Let now R be the ramification divisor of φ_V. Let $x \in X$. If $x \notin R$ then $|V - 2x|$ is empty, while if $x \in R$ then $|V - 2x|$ consists of a unique divisor D, corresponding to a hyperplane H, tangent to Δ at $\varphi_V(x)$. If $n = 2$ then D is clearly reducible as the union of two lines, thus $x \in \mathcal{R}(X, V)$ and $\mathcal{R}(X, V) = R = \mathcal{J}_1(X, V)$. If $n \geq 3$ then D is an irreducible quadric cone and therefore $\mathcal{R}(X, L) = \emptyset$.

Notice that $\varphi_{V|_D} : D \to H$ is a double cover ramified along the intersection of H and Δ, which is a reducible conic. The results of this discussion are summarized in the following table

n	$\mathrm{codim}_{|L|}(|V|)$	$\mathcal{R}(X, L)$
2	0	X
2	1	$\mathcal{J}_1(X, L)$
≥ 3	0, 1	$\emptyset$

EXAMPLE 5.12 (The second symmetric product of a hyperelliptic curve)**.** The following class of surfaces was considered in an apparently unrelated context in [5]. For the convenience of the reader we recall the construction. Let C_1 and C_2 be two copies of the same hyperelliptic curve of genus $q \geq 1$. Let $X_q = C_1 \times C_2$ and let $\pi_i : X \to C_i$, $i = 1, 2$, be the projections onto the factors. Let $\iota : X_q \to X_q$ be the involution $\iota(x, y) = (y, x)$ and let S be the quotient X_q/ι. Let $p : X_q \to S$ be the resulting double cover.

Choose a g_2^1 on C_i (pick the unique one if $q \geq 2$). Consider a divisor $D = Q_1 + Q_2 \in g_2^1$ Let $\hat{L}_i = \pi_i^* D$ and let $\hat{L} = \hat{L}_1 + \hat{L}_2$. It is $h^0(X_q, \hat{L}) = 4$. Let $H^0(X_q, \hat{L})^\iota$ be the subspace of global sections of $\hat{L}$ which are ι-invariant. If $H^0(C_i, D) = < \sigma, \tau >$ then $H^0(X_q, \hat{L})^\iota = < \sigma \otimes \sigma, \tau \otimes \tau, \sigma \otimes \tau + \tau \otimes \sigma >$. Now let $\hat{C} \in |\hat{L}|$ and consider the line bundle L on S associated to the divisor $p(\hat{C})$, so that $p^* L = \hat{L}$. There is a natural isomorphism between global sections of L and global sections of $\hat{L}$ which are ι-invariant. Therefore $h^0(S, L) = 3$. From the construction it follows that L is ample and spanned, and $2L^2 = \hat{L}^2 = (\hat{L}_1 + \hat{L}_2)^2 = 8$ so that $L^2 = 4$.

In [5], Section 5.2, it was shown that for all these surfaces $\mathrm{Bs}|K + L|$ contains a smooth rational component Γ. Moreover, for all $x \in \Gamma$ the pencil $|L - x|$ contains exactly one curve C singular at x, which is reducible, while every other member of the pencil is smooth at x and has at x the same tangent direction, see [5] Section 5.2 and Proposition 6.3. The latter result already shows that

$$(18) \qquad\qquad \Gamma \subseteq \mathcal{R}(S, L).$$

Now let $y_1, \ldots y_{2q+2}$ be the set of the ramification points of the chosen g_2^1 on C_i. These points, via the above construction, give rise to $(2q + 2)$ reducible elements of the form $2\Gamma_i \in |L|$.

Let $\mathfrak{J} = \{x_1, \ldots x_s\}$, $s = \binom{2q+2}{2}$, be the set of the intersection points of all pairs of $\Gamma_i's$. Notice that all the x_k's are distinct. If $x_k = \Gamma_r \cap \Gamma_s$, a simple check in local coordinates shows that every element in the pencil $|L - x_k|$ is singular at x_k but the only non reduced ones are precisely $2\Gamma_r$

and $2\Gamma_s$. This also shows that for all $x_k \in \mathfrak{J}$, it is $|L - x_k| = |L - 2x_k|$. Therefore it is $\mathfrak{J} \subseteq \mathcal{J}_2(S, L)$. Notice that the upper bound mentioned at the beginning of Section 3 gives $\binom{2q+2}{2} = \mathrm{Card}(\mathfrak{J}) \leq \mathrm{Card}(\mathcal{J}_2(S, L)) \leq 2q^2 + 3q + 3$ and thus $\mathfrak{J}$ misses at most two points of $\mathcal{J}_2(S, L)$. Notice also that

(19)   for all $x \in \mathcal{J}_2(S, L)$, it is $x \notin \mathcal{R}(S, L)$.

Otherwise (1) and the fact that $q(S) = h^1(\mathcal{O}_S) > 0$ would give $x \in \mathcal{B}(S, L)$ which contradicts Proposition 4.1.

Let now $x \in \Gamma_i \setminus \mathcal{J}_2(S, L)$ for some i. (Notice that this means $x \notin \mathfrak{J}$ and possibly x different from two more points.) As $x \notin \mathcal{J}_2(S, L)$ and $\dim(|L - x|) = 1$, it is $\dim(|L - 2x|) = 0$, i.e. the only singular element in $|L - x|$ is $2\Gamma_i$ which is reducible, and therefore

(20)   for all $x \in \Gamma_i \setminus \mathcal{J}_2(S, L)$ it is $x \in \mathcal{R}(S, L)$.

This shows that $\mathcal{W} = \Gamma \cup (\cup_{i=1}^{2q+2} \Gamma_i \setminus \mathcal{J}_2(S, L)) \subseteq \mathcal{R}(S, L)$. Recall from Lemma 5.1 that $\mathcal{R}(S, L) \subseteq \mathcal{J}_1(S, L)$. As $h^0(L) = 3$, φ_L gives a $4-1$ cover of $\mathbb{P}^2$ and thus the ramification locus of φ_L is a divisor, say R. Let $Supp(R)$ denote the reduced support of R. It is $Supp(R) = \mathcal{J}_1(S, L)$ and thus

(21) $\Gamma \cup (\cup_{i=1}^{2q+2} \Gamma_i) = \overline{\mathcal{W}} \subseteq \overline{\mathcal{R}(S, L)} \subseteq \mathcal{J}_1(S, L).$

We claim that R is reduced and that $Supp(R) = R = \mathcal{J}_1(S, L) = \Gamma + \Gamma_1 \ldots \Gamma_{2q+2}$ as divisors on S. To see this, first recall that R is linearly equivalent to $K_S + 3L$. If the claim were not true, (21) would imply that there exist positive integers ν, ν_i and an effective (or possibly trivial) divisor $\mathcal{D}$ such that

(22) $$\nu\Gamma + \sum_{i=1}^{2q+2} \nu_i \Gamma_i + \mathcal{D} \sim K + 3L.$$

The construction of these polarized surfaces shows that $g(L) = 2q$, hence $KL = 4q - 6$, and it also shows that $\Gamma L = 2$. Computing the intersection of both sides of the last equality with L we have

$$2(\nu + \sum \nu_i) + L\mathcal{D} = 2(q + 3).$$

As $\nu > 0$ and $\nu_i > 0$ for all i, and L is ample, the above equality implies $\nu = \nu_i = 1$ for all i and $\mathcal{D} = \mathcal{O}_X$. Therefore $Supp(R) = R = \mathcal{J}_1(S, L) = \Gamma \cup (\cup_{i=1}^{2q+2} \Gamma_i)$.

Now observe that $\varphi_{L|_\Gamma}$ is a $1 : 1$ map onto a smooth conic γ. To see this, notice that the only other possibility would be for $\varphi_{L|_\Gamma}$ to be a $2 : 1$ cover of a line. In this case, let z be a general point on

γ. Then $\varphi_{L_{|\Gamma}}^{-1}(z) = \{x, x'\}$ where $x, x' \in \Gamma, x \neq x'$. It follows that $|L - x| = |L - x'|$ but this contradicts the accurate description of $|L - x|$ given in [5], Proposition 6.3. In particular, the only singular element $D \in |L - x|$ is of the form $D = A + B$ with $A\Gamma = B\Gamma = 1$ and thus $D \notin |L - x'|$.

Further observe that $\varphi_{L_{|\Gamma_i}}$ is a $2 : 1$ map onto a line γ_i, tangent to γ, for all $i = 1, \ldots, 2q + 2$. To see this it is enough to recall that $L = 2\Gamma_i$ and that $\Gamma\Gamma_i = 1$ for all i.

The above discussion on the images of the components of the ramification divisor of φ_L shows that the branch locus of φ_L consists of the union of γ with $2q + 2$ tangent lines γ_i.

Then Lemma 2.2 and the fact that $\cup_i(\Gamma \cap \Gamma_i) \not\subset \mathcal{J}_2(S, L)$ gives

(23) $$\mathcal{J}_2(S, L) = \{x_1, \ldots, x_{2q+2}\}.$$

Finally, (18), (19), (20) and (23) give:

$$\mathcal{R}(S, L) = \Gamma \cup (\cup_{i=1}^{2q+2}\Gamma_i \setminus \{x_1, \ldots, x_{2q+2}\}).$$

EXAMPLE 5.13 (Double cover of $\mathbb{P}^1 \times \mathbb{P}^1$). Let $a \geq 2$ be an integer and let S be the smooth surface defined by the double cover $\pi : S \to \mathbb{Q}$ of the smooth quadric surface $\mathbb{Q} = \mathbb{P}^1 \times \mathbb{P}^1$ branched along a smooth curve $\Delta \in |\mathcal{O}_{\mathbb{Q}}(2a, 2a)|$. Let $L := \pi^*\mathcal{O}_{\mathbb{Q}}(1, 1)$. Then L is an ample and spanned line bundle on S. By using the projection formula we see that

$$h^0(L) = h^0(\pi_*L) = h^0(\mathcal{O}_{\mathbb{Q}}(1, 1) + h^0(\mathcal{O}_{\mathbb{Q}}(1 - a, 1 - a)) = 4.$$

This shows that the morphism $\varphi_L : S \to \mathbb{P}^3$ factors through π; in particular, L is not very ample. Let $x \notin \mathcal{J}_1(S, L)$. In this case $|L - 2x| = \{\pi^*h\}$, where h is the only element in $|\mathcal{O}_{\mathbb{Q}}(1, 1) - 2\pi(x)|$. Thus D is reducible, so being h, the section cut out on $\mathbb{Q}$ by its tangent plane at $\pi(x)$. Therefore $x \in \mathcal{R}(S, L)$. Now let $x \in \mathcal{J}_1(S, L)$. Recall that $\mathcal{J}_1(S, L)$ is the ramification curve of the double cover π, hence it is isomorphic to the branch curve Δ, via π. Thus

$$|L - 2x| = \{D = \pi^*h \quad | \quad h \in |\mathcal{O}_{\mathbb{Q}}(1, 1) - \tau_{\pi(x)}|\},$$

where $\tau_{\pi(x)}$ is a tangent vector to Δ at $\pi(x)$. Note that the general element h as above is irreducible, since it is cut out on $\mathbb{Q}$ by a plane of $\mathbb{P}^3$ not tangent to $\mathbb{Q}$ itself. Hence the corresponding element D in the pencil $|L - 2x|$ is also irreducible. This means that $x \notin \mathcal{R}(S, L)$. In conclusion we have:

$$\mathcal{R}(S, L) = S \setminus \mathcal{J}_1(S, L).$$

6. Towards the Conjecture

The examples in Section 5 lead us to formulate the following conjecture.

Conjecture 6.1. Let X be a smooth complex variety of dimension $n \geq 2$. Let L be an ample line bundle on X, spanned by a subspace $V \subseteq H^0(L)$ with $\dim(|V|) \geq n + 1$. If $\mathcal{R}(X, V) = X$ then (X, L) is either $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1))$ or a scroll over a smooth curve.

Conjecture 6.1 is very easily verified if one strengthens the requirements on $|V|$ by asking for its very ampleness. One can even relax a bit the condition on the size of $\mathcal{R}(X, V)$.

Proposition 6.2. *Let X be a smooth complex variety of dimension $n \geq 2$. Let L be a line bundle on X, with $V \subseteq H^0(L)$ such that $|V|$ is very ample and $\dim(|V|) \geq n + 1$. If $\mathcal{R}(X, V)$ is Zariski dense in X, then (X, L) is either $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1))$ or a scroll over a smooth curve.*

Proof. Consider X as embedded by $|V|$. Let x be a general point in X and thus $x \in \mathcal{R}(X, V)$. If $n = 2$ the general hyperplane tangent to X at x has no other tangency locus with X and thus it cuts on X a hyperplane section $D \in |L - 2x|$ that is reducible and has a single ordinary quadratic singularity at x. The proof now proceeds exactly as [3], Corollary 1.6.8, p. 31.

Let $n \geq 3$. For all $D \in |L - 2x|$, D is reducible and thus $\dim(\mathrm{Sing}(D)) \geq n - 2$. Notice that $\mathrm{Sing}(D)$ is the contact locus of the hyperplane corresponding to D and X. This being true for a general $x \in X$ implies that the dual defect of (X, V) is $\geq n - 2$. The conclusion now follows from [13], Corollary 3.4 or [6], Theorem 3.2.   □   □

Remark 6.3. Example 5.13 shows that the very ampleness is necessary if the size of $\mathcal{R}(X, V)$ is relaxed, as there $\mathcal{R}(X, L)$ is Zariski dense, L is ample and spanned but not very ample.

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